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A unified approach to the minimal unitary realizations of noncompact groups and supergroups

Murat Günaydin[a] and Oleksandr Pavlyk[ab]

[a] *Physics Department, Pennsylvania State University, University Park, PA 16802, U.S.A*
[b] *Wolfram Research Inc. 100 Trade Center Dr. Champaign, IL 61820, U.S.A.*
E-mail: murat@phys.psu.edu, pavlyk@phys.psu.edu

Abstract: We study the minimal unitary representations of non-compact groups and supergroups obtained by quantization of their geometric realizations as quasi-conformal groups and supergroups. The quasi-conformal groups G leave generalized light-cones defined by a quartic norm invariant and have maximal rank subgroups of the form $H\times \mathrm{SL}(2,\mathbb{R})$ such that $G/H\times \mathrm{SL}(2,\mathbb{R})$ are para-quaternionic symmetric spaces. We give a unified formulation of the minimal unitary representations of simple non-compact groups of type A_2, G_2, D_4,F_4, E_6, E_7, E_8 and $\mathrm{Sp}\,(2n,\mathbb{R})$. The minimal unitary representations of $\mathrm{Sp}\,(2n,\mathbb{R})$ are simply the singleton representations and correspond to a degenerate limit of the unified construction. The minimal unitary representations of the other noncompact groups $\mathrm{SU}\,(m,n)$, $\mathrm{SO}\,(m,n)$, $\mathrm{SO}^*(2n)$ and $\mathrm{SL}\,(m,\mathbb{R})$ are also given explicitly.
We extend our formalism to define and construct the corresponding minimal representations of non-compact supergroups G whose even subgroups are of the form $H\times \mathrm{SL}(2,\mathbb{R})$. If H is noncompact then the supergroup G does not admit any unitary representations, in general. The unified construction with H simple or Abelian leads to the minimal representations of $G(3), F(4)$ and $O\,\mathrm{Sp}\,(n|2,\mathbb{R})$ (in the degenerate limit). The minimal unitary representations of $O\,\mathrm{Sp}\,(n|2,\mathbb{R})$ with even subgroups $\mathrm{SO}(n)\times \mathrm{SL}(2,\mathbb{R})$ are the singleton representations. We also give the minimal realization of the one parameter family of Lie superalgebras $D\,(2,1;\sigma)$.

Keywords: Global Symmetries, Supergravity Models, Supersymmetry and Duality.

Contents

1. Introduction

Inspired by the work on spectrum generating Lie algebras by physicists [1] Joseph introduced the concept of minimal unitary realizations of Lie algebras. It is basically defined as a realization that exponentiates to a unitary representation of the corresponding noncompact group on a Hilbert space of functions depending on the minimal number of coordinates. Joseph gave the minimal realizations of the complex forms of classical Lie algebras and of G_2 in a Cartan-Weil basis [2, 3]. The existence of the minimal unitary representation of $E_{8(8)}$ within the framework of Langland's classification was first proved by Vogan [4] . Later, the minimal unitary representations of all simply laced groups, were studied by Kazhdan and Savin [5], and Brylinski and Kostant [6 – 9]. Gross and Wallach studied the minimal representations of quaternionic real forms of exceptional groups [10]. For a review and the references on earlier work on the subject in the mathematics literature prior to 2000 we refer the reader to the review lectures of Jian-Shu Li [11].

The idea that the theta series of $\mathrm{E}_{8(8)}$ and its subgroups may describe the quantum supermembrane in various dimensions [12], led Pioline, Kazhdan and Waldron [13] to reformulate the minimal unitary representations of simply laced groups [5]. In particular, they gave explicit realizations of the simple root (Chevalley) generators in terms of pseudo-differential operators for the simply laced exceptional groups, together with the spherical vectors necessary for the construction of modular forms.

Motivated mainly by the idea that the spectra of toroidally compactified M/superstring theories must fall into unitary representations of their U-duality groups and towards the goal of constructing these unitary representations Günaydin, Koepsell and Nicolai first studied the geometric realizations of U-duality groups of the corresponding supergravity theories [14]. In particular they gave geometric realizations of the U-duality groups of maximal supergravity in four and three dimensions as conformal and quasiconformal groups, respectively. The realization of the 3-dimensional U-duality group $E_{8(8)}$ of maximal supergravity given in [14] as a quasiconformal group that leaves invariant a generalized light-cone with respect to a quartic norm in 57 dimensions is the first known geometric realization of E_8. An $E_{7(7)}$ covariant construction of the minimal unitary representation of $E_{8(8)}$ by quantization of its geometric realization as a quasi-conformal group [14] was then given in [16]. The minimal unitary realization of the 3 dimensional U-duality group $E_{8(-24)}$ of the exceptional supergravity [27] by quantization of its geometric realization as a quasiconformal group was subsequently given in [15]. By consistent truncation the quasiconformal realizations of the other noncompact exceptional groups can be obtained from those of $E_{8(8)}$ and $E_{8(-24)}$. Apart from being the first known geometric realization of the exceptional group of type E_8 the quasiconformal realization has some remarkable features. First, there exist different real forms of all simple groups that admit realizations as quasiconformal groups.[1] Therefore, the quasiconformal realizations give a geometric meaning not only to the exceptional groups that appear in the last row of the famous Magic Square [28] but also extend to certain real forms of all simple groups.

Another remarkable property of the quasiconformal realizations is the above mentioned fact that their quantization leads, in a direct and simple manner, to the minimal unitary representations of the corresponding noncompact groups [16, 15, 17].

Classification of the orbits of the actions of U-duality groups on the BPS black hole solutions in maximal supergravity and $N = 2$ Maxwell-Einstein supergravity theories (MESGT) in five and four dimensions given in [30] suggested that four dimensional U-duality may act as a spectrum generating conformal symmetry in five dimensions [30, 14]. Furthermore, the work of [14] suggested that the 3 dimensional U duality group $E_{8(8)}$ of maximal supergravity must similarly act as a spectrum generating quasiconformal symme-

[1] For $\mathrm{SU}(1,1) = \mathrm{SL}(2,\mathbb{R}) = \mathrm{Sp}(2,\mathbb{R})$ the quasiconformal realization reduces to conformal realization.

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try group in the charge space of BPS black hole solutions in four dimensions extended by an extra coordinate which was interpreted as black hole entropy. This extends naturally to 3-dimensional U-duality groups of $N = 2$ MESGTs acting as spectrum generating quasiconformal symmetry groups in four dimensions [31]. More recently it was conjectured that the indexed degeneracies of certain $N = 8$ and $N = 4$ BPS black holes are given by some automorphic forms related to the minimal unitary representations of the corresponding 3 dimensional U-duality groups [32].

Motivated by the above mentioned results and conjectures stationary and spherically symmetric solutions of $N \geq 2$ supergravities with symmetric scalar manifolds were recently studied in [33]. By utilizing the equivalence of four dimensional attractor flow with the geodesic motion on the scalar manifold of the corresponding three dimensional theory the authors of [33] quantized the radial attractor flow, and argued that the three-dimensional U-duality groups must act as spectrum generating symmetry for BPS black hole degeneracies in 4 dimensions. They furthermore suggested that these degeneracies may be related to Fourier coefficients of certain modular forms of the 3-dimensional U-duality groups, in particular those associated with their minimal unitary representations.

The quasiconformal realizations of noncompact groups represent natural extensions of generalized conformal realizations of some of their subgroups and were studied from a spacetime point of view in [17]. The authors of [17] studied in detail the quasiconformal groups of generalized spacetimes defined by Jordan algebras of degree three. The generic Jordan family of Euclidean Jordan algebras of degree three describe extensions of the Minkowskian spacetimes by an extra "dilatonic" coordinate, whose rotation, Lorentz and conformal groups are $\mathrm{SO}(d-1), \mathrm{SO}(d-1,1) \times \mathrm{SO}(1,1)$ and $\mathrm{SO}(d,2) \times \mathrm{SO}(2,1)$, respectively. The generalized spacetimes described by simple Euclidean Jordan algebras of degree three correspond to extensions of Minkowskian spacetimes in the critical dimensions $(d = 3, 4, 6, 10)$ by a dilatonic *and* extra $(2, 4, 8, 16)$ commuting *spinorial* coordinates, respectively. Their rotation, Lorentz and conformal groups are those that occur in the first three rows of the Magic Square [28]. For the generic Jordan family the quasiconformal groups are $\mathrm{SO}(d + 2, 4)$. On the other hand, the quasiconformal groups of spacetimes defined by simple Euclidean Jordan algebras of degree are $\mathrm{F}_{4(4)}$, $\mathrm{E}_{6(2)}$, $\mathrm{E}_{7(-5)}$ and $\mathrm{E}_{8(-24)}$. The conformal subgroups of these quasiconformal groups are $\mathrm{Sp}(6, \mathbb{R}), SU^*(6), SO^*(12)$ and $E_{7(-25)}$, respectively.

In this paper we give a unified construction of the minimal unitary representations of noncompact groups by quantization of their geometric realizations as quasiconformal groups and extend it to the construction of the minimal representations of noncompact supergroups. In section 2 we explain the connection between minimal unitary representations of noncompact groups G and their unique para-quaternionic symmetric spaces of the form $G/H \times \mathrm{SL}(2, \mathbb{R})$, which was used in [21] to give a classification and minimal realizations of the real forms of infinite dimensional nonlinear quasi-superconformal Lie algebras that contain the Virasoro algebra as a subalgebra [20]. In section 3, using some of the results of [21] we give a unified construction of the minimal unitary representations of simple noncompact groups with H simple or Abelian. A degenerate limit of the unified construction leads to the minimal unitary representations of the symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$, which is discussed in section 4. In sections 5, 6 ,7 and 8 we give the minimal unitary realizations of $\mathrm{SO}(p+2,q+2)$, $SO^*(2n+4)$, $\mathrm{SU}(n+1,m+1)$ and $\mathrm{SL}(n+2,\mathbb{R})$, respectively. In section 9 we extend our construction to the minimal realizations of non-compact supergroups and present the unified construction of the minimal representations of supergroups whose even subgroups are of the form $H \times \mathrm{SL}(2,\mathbb{R})$ with H simple.The construction of the minimal unitary realizations of $O\,\mathrm{Sp}(N|2,\mathbb{R})$ corresponds to a degenerate limit of the unified construction and is discussed in section 10, where we also give the minimal realization of $D(2,1;\alpha)$. Preliminary results of sections 3 and 9 appeared in [29].

2. Minimal unitary representations of noncompact groups and para-quaternionic symmetric spaces

The minimal dimensions for simple non-compact groups were determined by Joseph [2, 3]. For a particular noncompact group G the minimal dimension ℓ can be found by considering the 5-graded decomposition of its Lie algebra $\mathfrak{g}$, determined by a distinguished $\mathfrak{sl}(2,\mathbb{R})$ subalgebra, of the form

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \left(\mathfrak{g}^{0} \oplus \Delta\right) \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} \tag{2.1}$$

where $\mathfrak{g}^{\pm 2}$ are 1-dimensional subspaces each, and Δ is the dilatation generator that determines the five grading. The generators belonging to the subspace $\mathfrak{g}^{-2} \oplus \Delta \oplus \mathfrak{g}^{+2}$ form the $\mathfrak{sl}(2\,,\mathbb{R})$ subalgebra in question. The minimal dimension ℓ is simply

$$\ell = \frac{1}{2} \dim\left(\mathfrak{g}^{+1}\right) + 1 \tag{2.2}$$

If we denote the subgroup generated by the grade zero subalgebra $\mathfrak{g}^{0}$ as H, then the quotient

$$\frac{G}{H \times \mathrm{SL}(2,\mathbb{R})} \tag{2.3}$$

is a para-quaternionic symmetric space in the terminology of [19]. Our goal in this paper is to complete the construction of the minimal unitary representations of all such non-compact groups by quantization of their quasiconformal realizations. Remarkably, the para-quaternionic symmetric spaces arose earlier in the classification [21] of infinite dimensional nonlinear quasi-superconformal Lie algebras that contain the Virasoro algebra as a subalgebra.[2] Below we list all simple noncompact groups G of this type and their subgroups H [21]:

[2]These infinite dimensional non-linear algebras were proposed as symmetry algebras that unify perturbative (Virasoro) and non-perturbative (U-duality) symmetries [22].

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G	H
$\mathrm{SU}(m,n)$	$\mathrm{U}(m-1,n-1)$
$\mathrm{SL}(n,\mathbb{R})$	$GL(n-2,\mathbb{R})$
$\mathrm{SO}(n,m)$	$\mathrm{SO}(n-2,m-2)\times\mathrm{SU}(1,1)$
$SO^*(2n)$	$SO^*(2n-4)\times\mathrm{SU}(2)$
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{Sp}(2n-2,\mathbb{R})$
$E_{6(6)}$	$\mathrm{SL}(6,\mathbb{R})$
$E_{6(2)}$	$\mathrm{SU}(3,3)$
$E_{6(-14)}$	$\mathrm{SU}(5,1)$
$E_{7(7)}$	$\mathrm{SO}(6,6)$
$E_{7(-5)}$	$SO^*(12)$
$E_{7(-25)}$	$\mathrm{SO}(10,2)$
$E_{8(8)}$	$E_{7(7)}$
$E_{8(-24)}$	$E_{7(-25)}$
$F_{4(4)}$	$\mathrm{Sp}(6,\mathbb{R})$
$G_{2(2)}$	$\mathrm{SU}(1,1)$

The minimal unitary representations of the exceptional groups (F_4, E_6, E_7, E_8) and of $\mathrm{SO}(n,4)$ as well as the corresponding quasiconformal realizations were given in [16, 15, 17].

3. Unified construction of the minimal unitary realizations of non-compact groups with H simple or abelian

Consider the 5-graded decomposition of the Lie algebra $\mathfrak{g}$ of G

$$\mathfrak{g}^{-2}\oplus\mathfrak{g}^{-1}\oplus\left(\mathfrak{g}^{0}\oplus\Delta\right)\oplus\mathfrak{g}^{+1}\oplus\mathfrak{g}^{+2}$$

Let J^a denote generators of the Lie algebra $\mathfrak{g}^0$ of H

$$\left[J^a\,,J^b\right]=f^{ab}{}_cJ^c \tag{3.1a}$$

where $a,b,\ldots=1,\ldots D$ and let ρ denote the symplectic representation by which $\mathfrak{g}^0$ acts on $\mathfrak{g}^{\pm1}$

$$[J^a\,,E^\alpha]=(\lambda^a)^\alpha{}_\beta E^\beta \qquad [J^a\,,F^\alpha]=(\lambda^a)^\alpha{}_\beta F^\beta \tag{3.1b}$$

where E^α, $\alpha,\beta,\ldots=1,\ldots,N=\dim(\rho)$ are generators that span the subspace $\mathfrak{g}^{-1}$

$$\left[E^\alpha\,,E^\beta\right]=2\Omega^{\alpha\beta}E \tag{3.1c}$$

and F^α are generators that span $\mathfrak{g}^{+1}$

$$\left[F^\alpha\,,F^\beta\right]=2\Omega^{\alpha\beta}F \tag{3.1d}$$

and $\Omega^{\alpha\beta}$ is the symplectic invariant "metric" of the representation ρ. The negative grade generators form a Heisenberg subalgebra since

$$[E^\alpha,E]=0 \tag{3.1e}$$

with the grade -2 generator E acting as its central charge. Similarly the positive grade generators form a Heisenberg algebra with the grade +2 generator F acting as its central charge. The remaining nonvanishing commutation relations of g are

$$\begin{aligned} F^\alpha &= [E^\alpha\,, F] \\ E^\alpha &= [E\,, F^\alpha] \\ \left[E^\alpha, F^\beta\right] &= -\Omega^{\alpha\beta}\Delta + \epsilon\lambda_a^{\alpha\beta}J^a \end{aligned} \qquad \begin{aligned} [\Delta, E^\alpha] &= -E^\alpha \\ [\Delta, F^\alpha] &= F^\alpha \\ [\Delta, E] &= -2E \\ [\Delta, F] &= 2F \end{aligned} \tag{3.1f}$$

where Δ is the generator that determines the five grading and ϵ is a parameter to be determined.

We shall realize the generators using bosonic oscillators ξ^α satisfying the canonical commutation relations

$$\left[\xi^\alpha\,, \xi^\beta\right] = \Omega^{\alpha\beta} \tag{3.2}$$

The grade -1, -2 generators and those of H can be realized easily as

$$E = \frac{1}{2}y^2 \qquad E^\alpha = y\,\xi^\alpha \qquad J^a = -\frac{1}{2}\lambda^a{}_{\alpha\beta}\xi^\alpha\xi^\beta \tag{3.3}$$

where y, at this point, is an extra "coordinate" such that $\frac{1}{2}y^2$ acts as the central charge of the Heisenberg algebra formed by the negative grade generators.

Now there may exist different real forms of G with different subgroups H. For reasons that will become obvious we shall assume that a real form of G exists for which H is simple. We shall follow the conventions of [21] throughout this paper except for the occasional use of Cartan labeling of simple Lie algebras whenever we are not considering specific real forms.

The quadratic Casimir operator of the Lie algebra $\mathfrak{g}^0$ of H is

$$\mathcal{C}_2\left(\mathfrak{g}^0\right) = \eta_{ab}J^aJ^b \tag{3.4}$$

where η_{ab} is the Killing metric of H. The minimal realizations given in [16, 15, 17] and the results of [21] suggest an Ansatz for the grade +2 generator F of the form

$$F = \frac{1}{2}p^2 + \kappa y^{-2}\left(\mathcal{C}_2 + \mathfrak{C}\right) \tag{3.5}$$

where p is the momentum conjugate to the coordinate y

$$[y, p] = i \tag{3.6}$$

and κ and $\mathfrak{C}$ are some constants to be determined later. This implies then

$$\begin{aligned} F^\alpha &= [E^\alpha, F] = ip\,\xi^\alpha + \kappa y^{-1}\left[\xi^\alpha\,, \mathcal{C}_2\right] \\ &= ip\,\xi^\alpha - \kappa y^{-1}\left[2\left(\lambda^a\right)^\alpha{}_\beta\xi^\beta J_a + C_\rho\,\xi^\alpha\right] \end{aligned} \tag{3.7}$$

where C_ρ is the eigenvalue of the second order Casimir of H in the representation ρ.[3]

[3]Note that the indices $\alpha, \beta, \ldots$ are raised and lowered with the antisymmetric symplectic metric $\Omega^{\alpha\beta} = -\Omega^{\beta\alpha}$ that satisfies $\Omega^{\alpha\beta}\Omega_{\gamma\beta} = \delta^\alpha_\beta$ and $V^\alpha = \Omega^{\alpha\beta}V_\beta$, and $V_\alpha = V^\beta\Omega_{\beta\alpha}$. In particular, we have $V^\alpha W_\alpha = -V_\alpha W^\alpha$.

We choose the normalization of the representation matrices λ as in [20, 21]

$$\lambda^{a,\alpha\beta}\lambda_{a\ \delta}^{\ \gamma} - \lambda^{a,\gamma\alpha}\lambda_{a\ \delta}^{\ \beta} = -\frac{C_\rho}{N+1}\left(\Omega^{\alpha\beta}\delta^{\gamma}_{\ \delta} - 2\Omega^{\beta\gamma}\delta^{\alpha}_{\ \delta} + \Omega^{\gamma\alpha}\delta^{\beta}_{\ \delta}\right), \tag{3.8}$$

which implies

$$\lambda^a_{\alpha\beta}\lambda^{\beta\gamma}_a = -C_\rho\delta^\gamma_\alpha \tag{3.9}$$

The unknown constants in our Ansatz will be determined by requiring that generators satisfy the commutation relations (3.1) of the Lie algebra $\mathfrak{g}$. We first consider commutators of elements of $\mathfrak{g}^1$ and $\mathfrak{g}^{-1}$

$$\left[E^\alpha , F^\beta\right] = i\,(y\,p)\,\Omega^{\alpha\beta} - \xi^\beta\,\xi^\alpha + \kappa\left[\xi^\alpha , \left[\xi^\beta , \mathcal{C}_2\right]\right] \tag{3.10}$$

which, upon using the identity,

$$\left[\xi^\alpha , \mathcal{C}_2\right] = -2\,(\lambda^a)^\alpha_{\ \beta}\xi^\beta J_a - C_\rho\,\xi^\alpha \tag{3.11}$$

leads to

$$\left[E^\alpha , F^\beta\right] = -\Delta\Omega^{\alpha\beta} + \left\{\frac{3\kappa C_\rho}{1+N} - \frac{1}{2}\right\}\left(\xi^\alpha\xi^\beta + \xi^\beta\xi^\alpha\right) - 6\kappa\,(\lambda^a)^{\alpha\beta}\,J_a \tag{3.12}$$

where $\Delta = -\frac{i}{2}\,(yp + py)$. Now the bilinears $\left(\xi^\alpha\xi^\beta + \xi^\beta\xi^\alpha\right)$ generate the Lie algebra of $\mathfrak{c}_{N/2}\,(\mathfrak{sp}\,(N))$ under commutation. Hence for those Lie algebras $\mathfrak{g}$ whose subalgebras $\mathfrak{g}^0$ are different from $\mathfrak{c}_{N/2}$ closure requires that the coefficient of the second term vanish

$$\frac{3\kappa C_\rho}{1+N} - \frac{1}{2} = 0 \tag{3.13}$$

For Lie algebras $\mathfrak{g}$ whose subalgebras $\mathfrak{g}^0$ are of type $\mathfrak{c}_{N/2}$ we have

$$(\lambda_a)^{\alpha\beta}\,J^a \approx \xi^\alpha\xi^\beta + \xi^\beta\xi^\alpha$$

Hence we do not get any constraints on κ from the above commutation relation.

Next, let us compute the commutator

$$\left[F^\alpha , F^\beta\right] = \frac{\kappa}{y^2}\left(-\xi^\alpha\left[\xi^\beta , \mathcal{C}_2\right] + \xi^\beta\left[\xi^\alpha , \mathcal{C}_2\right] + \kappa\left[\left[\xi^\alpha , \mathcal{C}_2\right], \left[\xi^\beta , \mathcal{C}_2\right]\right]\right) - p^2\Omega^{\alpha\beta} \tag{3.14}$$

Using (3.8), (3.11) we write the terms linear in κ on the right hand side as

$$\frac{\kappa}{y^2}\left(-\xi^\alpha\left[\xi^\beta , \mathcal{C}_2\right] + \xi^\beta\left[\xi^\alpha , \mathcal{C}_2\right]\right) = \frac{\kappa}{y^2}\left(C_\rho\Omega^{\alpha\beta} + 2\left(\xi^\alpha(\lambda^a)^\beta_{\ \gamma} - \xi^\beta(\lambda^a)^\alpha_{\ \gamma}\right)\xi^\gamma J_a\right). \tag{3.15}$$

The terms quadratic in κ on the right hand side gives

$$\kappa\left[\left[\xi^\alpha , \mathcal{C}_2\right], \left[\xi^\beta , \mathcal{C}_2\right]\right] = \frac{12\kappa C_\rho}{N+1}\left(\xi^\alpha(\lambda^a)^\beta_{\ \gamma} - \xi^\beta(\lambda^a)^\alpha_{\ \gamma}\right)\xi^\gamma J_a - \kappa C_\rho^2\Omega^{\alpha\beta}$$
$$+4\kappa\left(3\left(\lambda^b\lambda^a\right)^{\alpha\beta} J_a J_b - 2\left(\lambda^b\lambda^a\right)^{\beta\alpha} J_a J_b + f_{ab}{}^c(\lambda^a)^\alpha_{\ \mu}\left(\lambda^b\right)^\beta_{\ \nu}\xi^\mu\xi^\nu J_c\right)$$

Using these two expressions above (3.14) becomes

$$\begin{aligned}\left[F^{\alpha}\,,F^{\beta}\right] &= -2\left(\frac{1}{2}p^2+\frac{1}{y^2}\left(\frac{\kappa^2}{2}C_\rho^2-\frac{\kappa}{2}C_\rho\right)\right)\Omega^{\alpha\beta} \\ &+\frac{4\kappa}{y^2}\left(\xi^\alpha(\lambda^a)^\beta{}_\gamma-\xi^\beta(\lambda^a)^\alpha{}_\gamma\right)\xi^\gamma J_a \\ &+\frac{4\kappa^2}{y^2}\left(3\left(\lambda^b\lambda^a\right)^{\alpha\beta}J_aJ_b-2\left(\lambda^b\lambda^a\right)^{\beta\alpha}J_aJ_b+f_{ab}{}^c(\lambda^a)^\alpha{}_\mu\left(\lambda^b\right)^\beta{}_\nu\xi^\mu\xi^\nu J_c\right)\end{aligned} \tag{3.16}$$

Now the right hand side of (3.16) must equal $2\Omega^{\alpha\beta}F$ with

$$F=\frac{1}{2}p^2+\kappa y^{-2}\left(\mathcal{C}_2+\mathfrak{C}\right)$$

per our Ansatz. Contracting the right hand side of (3.16) with $\Omega_{\beta\alpha}$ we get

$$-N\left(p^2+\frac{1}{y^2}\left(\kappa^2C_\rho^2-\kappa C_\rho\right)\right)-\frac{1}{y^2}\kappa\left(-16+20\kappa i_\rho\ell^2-4\kappa C_{\rm adj}\right)\mathcal{C}_2 \tag{3.17}$$

where i_ρ is the Dynkin index of the representation ρ of H and C_{adj} is the eigenvalue of the second order Casimir in the adjoint of H. To obtain this result one uses the fact that

$$\lambda^a_{\alpha\beta}\lambda^{b,\alpha\beta}=-i_\rho\ell^2\eta^{ab}$$

where ℓ is the length of the longest root of H.[4] Using

$$C_{adj}=-\ell^2h^\vee \tag{3.18}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}^0$ subalgebra of $\mathfrak{g}$, the closure then requires

$$\left(-8+10\kappa i_\rho\ell^2+2\kappa h^\vee\ell^2\right)=N \tag{3.19}$$

Equations (3.13) and (3.19), combined with

$$i_\rho\ell^2=\frac{N}{D}C_\rho \tag{3.20}$$

imply

$$\frac{h^\vee}{i_\rho}=\frac{3D}{N(N+1)}(N+8)-5 \tag{3.21}$$

The validity of the above expression can be verified explicitly by comparing with table 1 of [21], relevant part of which is collected in table 1 for convenience. Furthermore, it was shown in [21] that all the groups and the corresponding symplectic representations listed in the above table satisfy the equation

$$h^\vee=2i_\rho\left(\frac{D}{N}+\frac{3D}{N(1+N)}-1\right) \tag{3.22}$$

[4] The length squared ℓ^2 of the longest root is normalized such that it is 2 for the simply laced algebras, 4 for B_n, C_n and F_4 and 6 for G_2. The i_ρ, C_ρ and ℓ are related by $i_\rho=\frac{NC_\rho}{D\ell^2}$ where $D=\dim(H)=\dim(g^0)$.

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g^0	D	$h^\vee$	$N = dim\rho$	i_ρ
$\mathfrak{c}_n$	$n(2n+1)$	$n+1$	$2n$	$\frac{1}{2}$
$\mathfrak{a}_5$	35	6	20	3
$\mathfrak{d}_6$	66	10	32	4
$\mathfrak{e}_7$	133	18	56	6
$\mathfrak{c}_3$	21	4	14	$\frac{5}{2}$
$\mathfrak{a}_1$	3	2	4	5

Table 1: The list of grade zero subalgebras g^0 with dual Coxeter number $h^\vee$ that are simple and with irreducible action ρ on grade +1 subspace. i_ρ is the Dynkin index of the representation ρ.

which was obtained as a consistency condition for the existence of certain class of infinite dimensional nonlinear quasi-superconformal algebras. Comparing this equation with the equation (3.21) we see that they are consistent with each other if

$$D = \frac{3N\,(N+1)}{N+16} \tag{3.23}$$

Requirement of $[F\,, F^\alpha] = 0$ leads to the condition

$$\xi^\alpha\,(\mathcal{C}_2 + \mathfrak{C}) + (\mathcal{C}_2 + \mathfrak{C})\,\xi^\alpha + \kappa\,[\mathcal{C}_2\,, [\xi^\alpha\,, \mathcal{C}_2]] = 0 \tag{3.24}$$

Using (3.11) and $[\mathcal{C}_2\,, J^a] = 0$ we arrive at

$$\begin{aligned} 2\,\xi^\alpha\,(\mathcal{C}_2 + \mathfrak{C}) + 2\,(1-\kappa C_\rho)\,C_\rho \xi^\alpha + 2\,(1-\kappa C_\rho)\,(\lambda^a)^\alpha{}_\beta\,\xi^\beta J_a \\ -4\,\kappa\left(\lambda^a\lambda^b\right)^\alpha{}_\beta\,\xi^\beta J_b J_a = 0 \end{aligned} \tag{3.25}$$

In order to extract restrictions on $\mathfrak{g}$ implied by the above equation we contract it with $\xi^\gamma\Omega_{\gamma\alpha}$ and obtain

$$\frac{h^\vee}{i_\rho} = \frac{D}{N(N+1)}(N-8) + 1\,. \tag{3.26}$$

It agrees with (3.21) provided (3.23) holds true. Making use of

$$N = 2(g^\vee - 2)$$

where $g^\vee$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$ and (3.23) we obtain

$$\dim(\mathfrak{g}) = 2 + 2N + 1 + \dim\left(\mathfrak{g}^0\right) = 1 + 2\,(N+1) + D = 2\frac{(g^\vee+1)\;(5g^\vee-6)}{g^\vee+6} \tag{3.27}$$

Equation (3.24) and the requirement of the right hand side of (3.16) to equal to $2\Omega^{\alpha\beta}F$ imply restriction on matrices λ^a for which (3.21) and (3.26) are only necessary conditions. We expect these conditions to be derivable from the identities satisfied by the corresponding Freudenthal triple systems [36] that underlie the quasiconformal actions and the minimal realizations [14, 17].

By going through the list of simple Lie algebras [25] collected for convenience in table 2 we see that the equation (3.23) is valid only for the Lie algebras of simple groups A_1, A_2,

$\mathfrak{g}$	$\dim(\mathfrak{g})$	$g^\vee$	Eqtn. (3.27) holds ?
$\mathfrak{a}_n$	n^2+2n	$n+1$	for $\mathfrak{a}_1$ and $\mathfrak{a}_2$ only
$\mathfrak{b}_n$	$2n^2+n$	$2n-1$	no
$\mathfrak{c}_n$	$2n^2+n$	$n+1$	no
$\mathfrak{d}_n$	$2n^2-n$	$2n-2$	for $\mathfrak{d}_4$ only
$\mathfrak{e}_6$	78	12	yes
$\mathfrak{e}_7$	133	18	yes
$\mathfrak{e}_8$	248	30	yes
$\mathfrak{f}_4$	52	9	yes
$\mathfrak{g}_2$	14	4	yes

Table 2: Dimensions and dual Coxeter numbers of simple Lie algebras. In order for the Lie algebra to admit a non-trivial 5-graded decomposition its dimension must be greater than 6. This rules out $\mathfrak{sl}(2)$ for which (3.27) also holds.

G_2, D_4, F_4, E_6, E_7 and E_8. For A_1 our realization reduces simply to the conformal realization. With the exception of D_4, what these groups have in common is the fact that their subgroups H are either simple or one dimensional Abelian as expected by the consistency with our Ansatz. The reason our Ansatz also covers the case of D_4 has to do with its unique properties. The subalgebra $\mathfrak{g}^0$ of $\mathfrak{d}_4$ is the direct sum of three copies of $\mathfrak{a}_1$

$$\mathfrak{g}^0 = \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1$$

The eigenvalues of the quadratic Casimirs of these subalgebras $\mathfrak{a}_1$ as well as their Dynkin indices in the representation ρ coincide as required by the consistency with our Ansatz. These groups appear in the last row of the so-called Magic Triangle [26] which extends the Magic Square of Freudenthal, Rozenfeld and Tits [28].

There is, in addition, an infinite family of non-compact groups for which H is simple, namely the noncompact symplectic groups $\mathrm{Sp}(2n+2,\mathbb{R})$ with $\dim\rho = 2n$ and $H = \mathrm{Sp}(2n,\mathbb{R})$. However, as remarked above, the constraint (3.13) and hence the equation (3.23) do not follow from our Ansatz for the symplectic groups. The quartic invariant becomes degenerate for symplectic groups and the minimal unitary realizations reduce to free boson construction of the singleton representations for these groups as will be discussed in the next section.

The minimal unitary realizations of noncompact groups appearing in the Magic Triangle [28, 26] can be obtained by consistent truncation of the minimal unitary realizations of the groups appearing in its last row [15, 14, 17]. We should stress that there are different real forms of the groups appearing in the Magic Square or its straightforward extension to the Magic Triangle. Different real forms in our unified construction correspond to different hermiticity conditions on the bosonic oscillators ξ^α [15]. After specifying the hermiticity properties of the oscillators ξ^α one goes to a Hermitian (anti-hermitian) basis of the Lie algebra $\mathfrak{g}$ with purely imaginary (real) structure constants to calculate the Killing metric which determines the real form corresponding to the minimal unitary realization.

The quadratic Casimir operator of the Lie algebra constructed in a unified manner above is given by

$$\mathcal{C}_2(\mathfrak{g}) = J^a J_a + \frac{2\,C_\rho}{N+1}\left(\frac{1}{2}\Delta^2 + EF + FE\right) - \frac{C_\rho}{N+1}\Omega_{\alpha\beta}\left(E^\alpha F^\beta + F^\beta E^\alpha\right) \tag{3.28}$$

which, upon using (3.13) and the following identities

$$\begin{aligned} \frac{1}{2}\Delta^2 + EF + FE &= \kappa\left(J^a J_a + \mathfrak{C}\right) - \frac{3}{8} \\ \Omega_{\alpha\beta}\left(E^\alpha F^\beta + F^\beta E^\alpha\right) &= 8\,\kappa J^a J_a + \frac{N}{2} + \kappa C_\rho N \end{aligned} \tag{3.29}$$

that follow from our Ansatz, reduces to a c-number

$$\begin{aligned} \mathcal{C}_2(\mathfrak{g}) &= \mathfrak{C}\left(\frac{8\kappa C_\rho}{N+1} - 1\right) - \frac{3}{4}\frac{C_\rho}{N+1} - \frac{N}{2}\frac{C_\rho}{N+1} - \frac{\kappa C_\rho^2 N}{N+1} \\ &=_{(\text{using eq.(3.13)})} -\frac{C_\rho}{36}\frac{(N+4)(5\,N+8)}{N+1} \end{aligned} \tag{3.30}$$

as required by irreducibility. We should note that this result agrees with explicit calculations for the Lie algebras of the Magic Square in [15]. In the normalization chosen there $\kappa = 2$ and hence $12C_\rho = N+1$. Then, using $N = 2g^\vee - 4$ we get

$$\mathcal{C}_2(\mathfrak{g}) = -\frac{1}{108}\left(5g^\vee - 6\right)g^\vee. \tag{3.31}$$

4. The minimal unitary representations of $Sp(2n+2,\mathbb{R})$

The Lie algebra of $Sp(2n+2,\mathbb{R})$ has a 5-grading of the form

$$\mathfrak{sp}(2n+2,\mathbb{R}) = E \oplus E^\alpha \oplus \left(\mathfrak{sp}(2n,\mathbb{R}) \oplus \Delta\right) \oplus F^\alpha \oplus F \tag{4.1}$$

where $E^\alpha = y\xi^\alpha$, and $E = \frac{1}{2}y^2$. Generators of the grade zero subalgebra $\mathfrak{g}^0 = \mathfrak{sp}(2n,\mathbb{R})$ are given simply by the symmetrized bilinears (modulo normalization)

$$-2\left(\lambda_a\right)^{\alpha\beta} J^a = \xi^\alpha\xi^\beta + \xi^\beta\xi^\alpha \tag{4.2}$$

which is simply the singleton (metaplectic) realization of $\mathfrak{sp}(2n,\mathbb{R})$. The quadratic Casimir of $\mathfrak{sp}(2n,\mathbb{R})$ in the singleton realizaton is simply a c-number. As stated in the previous section the constraint equation (3.13) that follow from the commutation relations $[E^\alpha, F^\beta]$ can not be imposed in the case of symplectic Lie algebras $Sp(2n+2,\mathbb{R})$. However the equation (3.24) that follows from our Ansatz requires that $\mathcal{C}_2 + \mathfrak{C} = 0$ or $\kappa = 0$ for the symplectic Lie algebras $Sp(2n+2,\mathbb{R})$. In other words $F = \frac{1}{2}p^2$. Thus

$$F^\alpha = \left[E^\alpha, F\right] = ip\xi^\alpha \tag{4.3}$$

$$\begin{aligned} \left[E^\alpha, F^\beta\right] &= i\,(yp)\,\Omega^{\alpha\beta} - \xi^\beta\xi^\alpha \\ &= -\frac{i}{2}\Delta - \frac{1}{2}(\xi^\alpha\xi^\beta + \xi^\beta\xi^\alpha) \end{aligned} \tag{4.4}$$

Thus the minimal unitary realization of the symplectic group $\mathrm{Sp}(2n+2,\mathbb{R})$ obtained by quantization of its quasiconformal realization is simply the singleton realization in terms bilinears of the $2n+2$ oscillators (annihilation and creation operators) (ξ^α, y, p). The quadratic Casimir of $\mathfrak{sp}\,(2n+2,\mathbb{R})$ is also a c-number.

That the quadratic Casimir is a c-number is only a necessary requirement for the irreducibility of the corresponding represenation. For the above singleton realization the entire Fock space of all the oscillators decompose into the direct sum of the two inequivalent singleton representations that have the same eigenvalue of the quadratic Casimir. They are both unitary lowest weight representations. By choosing a definite polarization one can define $n+1$ annihilation operators

$$a_0 = \frac{1}{\sqrt{2}}(y + ip)$$

$$a_i = \frac{1}{\sqrt{2}}(\xi^i + i\xi^{n+i}) \qquad i = 1, 2, \ldots, n$$

and $n+1$ creation operators

$$a^0 = \frac{1}{\sqrt{2}}(y - ip)$$

$$a^i = \frac{1}{\sqrt{2}}(\xi^i - i\xi^{n+i})$$

in terms of the (n+1) coordinates and (n+1) momenta. The vacuum vector $|0\rangle$ annihilated by all the annihilation operators

$$a_0|0\rangle = a_i|0\rangle = 0$$

is the lowest weight vector of the "scalar" singleton irrep of $\mathrm{Sp}(2n+2,\mathbb{R})$ and (n+1) vectors

$$a^0|0\rangle, a^i|0\rangle$$

form the lowest K-vector of the other singleton irrep of $\mathrm{Sp}(2n+2,\mathbb{R})$. In other words the lowest K vector of the scalar singleton is an $\mathrm{SU}(n+1)$ scalar, while the lowest K-vector of the other singleton irrep is a vector of $\mathrm{SU}(n+1)$ subgroup of $\mathrm{Sp}(2n+2,\mathbb{R})$. Both lowest K-vectors carry a nonzero $\mathrm{U}(1)$ charge.

The reason for the reduction of the minimal unitary realizations of the Lie algebras of symplectic groups $\mathrm{Sp}(2n+2,\mathbb{R})$ to bilinears, and hence to a free boson construction, is the fact that there do not exist any nontrivial quartic invariant of $\mathrm{Sp}(2n,\mathbb{R})$ defined by an irreducible symmetric tensor in the fundamental representation $2n$. We have only the skew symmetric symplectic invariant tensor $\Omega\alpha\beta$ in the fundamental representation, which when contracted with $\xi^\alpha\xi^\beta$ gives a c-number.

In the light of the above results one may wonder how the quasi-conformal realization of $\mathfrak{sp}\,(2n+2,\mathbb{R})$ can be made manifest. Before quantisation we have $2n+1$ coordinates $\mathbb{X} = (X^\alpha, x)$ on which we realize $\mathfrak{sp}\,(2n+2,\mathbb{R})$:

$$\mathcal{N}\left(\mathbb{X}\right) = I_4\left(X^\alpha\right) - x^2 \tag{4.5}$$

since $I_4(X^\alpha) = 0$. With the "twisted" difference vector defined as [14]

$$\delta(\mathbb{X}, \mathbb{Y}) = (X^\alpha - Y^\alpha, x - y + \langle X, Y \rangle) \tag{4.6}$$

The equation defining the generalized lightcone

$$\mathcal{N}(\delta(\mathbb{X}, \mathbb{Y})) = 0$$

then reduces to

$$x - y + \langle X, Y \rangle = 0 \tag{4.7}$$

where $\langle X, Y \rangle = \Omega_{\alpha\beta} X^\alpha Y^\beta$. By reinterpreting the coordinates (X^α, x) and (Y^α, y) as projective coordinates in 2n+2 dimensional space

$$x = \frac{\xi^0}{\xi^{n+1}}$$

$$X^\alpha = \frac{\xi^\alpha}{\xi^{n+1}}$$

$$y = \frac{\eta^0}{\eta^{n+1}}$$

$$Y^\alpha = \frac{\eta^\alpha}{\eta^{n+1}}$$

the above equation for the light cone can be written in the form

$$\xi^0 \eta^{n+1} - \eta^0 \xi^{n+1} + \Omega_{\alpha\beta} \xi^\alpha \eta^\beta = 0$$

which is manifestly invariant under $\mathrm{Sp}(2n+2, \mathbb{R})$.

5. Minimal unitary realizations of the quasiconformal groups $\mathrm{SO}(p+2, q+2)$

In our earlier work [17] we constructed the minimal unitary representations of $SO(d+2, 4)$ obtained by quantization of their realizations as quasiconformal groups. That construction carries over in a straightforward manner to the other real forms $SO(p+2, q+2)$ which we give in this section. They were studied also in [34] using the quasiconformal approach and in [35] by other methods.

Now the relevant subgroup for the minimal unitary realization is

$$\mathrm{SO}(p, q) \times \mathrm{SO}(2, 2) \subset \mathrm{SO}(p+2, q+2) \tag{5.1}$$

where

$$\mathrm{SO}(2, 2) = Sl(2, \mathbb{R}) \times Sl(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R}) \times \mathrm{Sp}(2, \mathbb{R}) \tag{5.2}$$

and one of factors above can be identified with the distinguished $Sl(2, \mathbb{R})$ subgroup. The relevant 5-grading of the Lie algebra of $\mathrm{SO}(p+2, q+2)$ is then given as

$$\mathfrak{so}(p+2, q+2) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus (\mathfrak{so}(p, q) \oplus \mathfrak{sp}(2, \mathbb{R}) \oplus \Delta) \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} \tag{5.3}$$

where grade ± 1 subspaces transform in the $[(p+q),2]$ dimensional representation of $\mathrm{SO}(p,q)\times Sl(2,\mathbb{R})$.

Let X^μ and P_μ be canonical coordinates and momenta in $\mathbb{R}^{(p,q)}$:

$$[X^\mu, P_\nu] = i\delta^\mu_\nu \tag{5.4}$$

Also let x be an additional "cocycle" coordinate and p be its conjugate momentum:

$$[x,p] = i \tag{5.5}$$

They are taken to satisfy the following Hermiticity conditions:

$$(X^\mu)^\dagger = \eta_{\mu\nu}X^\nu \qquad (P_\mu)^\dagger = \eta^{\mu\nu}P_\nu \qquad p^\dagger = p \qquad x^\dagger = x \tag{5.6}$$

where $\eta_{\mu\nu}$ is the $\mathrm{SO}(p,q)$ invariant metric. The subgroup H of $\mathrm{SO}(p+2,q+2)$ is $\mathrm{SO}(p,q)\times \mathrm{Sp}(2,\mathbb{R})_J$ whose generators we will denote as $(M_{\mu\nu},\, J_\pm,\, J_0)$. The grade -1 generators will be denoted as $(U_\mu,\, V^\mu)$ and the grade -2 generator as K_-. The generators of H, its 4-th order invariant $\mathcal{I}_4$ and the negative grade generators are realized as follows:

$$\begin{aligned}
M_{\mu\nu} &= i\eta_{\mu\rho}X^\rho P_\nu - i\eta_{\nu\rho}X^\rho P_\mu & J_0 &= \frac{1}{2}\left(X^\mu P_\mu + P_\mu X^\mu\right) \\
U_\mu &= xP_\mu \qquad V^\mu = xX^\mu & J_- &= X^\mu X^\nu \eta_{\mu\nu} \\
K_- &= \frac{1}{2}x^2 & J_+ &= P_\mu P_\nu \eta^{\mu\nu} \\
\mathcal{I}_4 &= \left(X^\mu X^\nu \eta_{\mu\nu}\right)\left(P_\mu P_\nu \eta^{\mu\nu}\right) + \left(P_\mu P_\nu \eta^{\mu\nu}\right)\left(X^\mu X^\nu \eta_{\mu\nu}\right) \\
&\quad - \left(X^\mu P_\mu\right)\left(P_\nu X^\nu\right) - \left(P_\mu X^\mu\right)\left(X^\nu P_\nu\right)
\end{aligned} \tag{5.7}$$

where $\eta_{\mu\nu}$ is the flat metric with signature (p,q).

It is easy to verify that the generators $M_{\mu\nu}$ and $J_{0,\pm}$ satisfy the commutation relations of $\mathfrak{so}\,(p,q)\oplus\mathfrak{sp}\,(2,\mathbb{R})$

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\tau}] &= \eta_{\nu\rho}M_{\mu\tau} - \eta_{\mu\rho}M_{\nu\tau} + \eta_{\mu\tau}M_{\nu\rho} - \eta_{\nu\tau}M_{\mu\rho} \\
[J_0, J_\pm] &= \pm 2iJ_\pm \qquad [J_-, J_+] = 4iJ_0
\end{aligned} \tag{5.8}$$

under which coordinates X^μ and momenta P_μ transform as $SO\,(p,q)$ vectors and form doublets of the symplectic group $\mathrm{Sp}(2,\mathbb{R})_J$:

$$\begin{aligned}
[J_0, V^\mu] &= -iV^\mu \quad & [J_-, V^\mu] &= 0 \quad & [J_+, V^\mu] &= -2i\eta^{\mu\nu}U_\nu \\
[J_0, U_\mu] &= +iU_\mu \quad & [J_-, U_\mu] &= 2i\eta_{\mu\nu}V^\nu \quad & [J_+, U_\mu] &= 0
\end{aligned} \tag{5.9}$$

The generators in the subspace $\mathfrak{g}^{-1}\oplus\mathfrak{g}^{-2}$ form a Heisenberg algebra

$$[V^\mu, U_\nu] = 2i\delta^\mu{}_\nu\, K_- \,. \tag{5.10}$$

with K_- playing the role of central charge.

Using the quartic invariant we define the grade $+2$ generator as

$$K_+ = \frac{1}{2}p^2 + \frac{1}{4\,y^2}\left(\mathcal{I}_4 + \frac{(p+q-2)^2+3}{2}\right) \tag{5.11}$$

Then the grade +1 generators are obtained by commutation relations

$$\tilde{V}^{\mu} = -i\left[V^{\mu}, K_{+}\right] \qquad \tilde{U}_{\mu} = -i\left[U_{\mu}, K_{+}\right] \tag{5.12}$$

which explicitly read as follows

$$\begin{aligned}
\tilde{V}^{\mu} &= pX^{\mu} + \frac{1}{2}x^{-1}\left(P_{\nu}X^{\lambda}X^{\rho} + X^{\lambda}X^{\rho}P_{\nu}\right)\eta^{\mu\nu}\eta_{\lambda\rho} \\
&\quad - \frac{1}{4}x^{-1}\left(X^{\mu}\left(X^{\nu}P_{\nu} + P_{\nu}X^{\nu}\right) + \left(X^{\nu}P_{\nu} + P_{\nu}X^{\nu}\right)X^{\mu}\right) \\
\tilde{U}_{\mu} &= pP_{\mu} - \frac{1}{2}x^{-1}\left(X^{\nu}P_{\lambda}P_{\rho} + P_{\lambda}P_{\rho}X^{\nu}\right)\eta_{\mu\nu}\eta^{\lambda\rho} \\
&\quad + \frac{1}{4}x^{-1}\left(P_{\mu}\left(X^{\nu}P_{\nu} + P_{\nu}X^{\nu}\right) + \left(X^{\nu}P_{\nu} + P_{\nu}X^{\nu}\right)P_{\mu}\right).
\end{aligned} \tag{5.13}$$

Then one finds that the generators in $\mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}$ subspace form an Heisenberg algebra as well

$$\left[\tilde{V}^{\mu}, \tilde{U}_{\nu}\right] = 2i\delta^{\mu}{}_{\nu}K_{+} \qquad V^{\mu} = i\left[\tilde{V}^{\mu}, K_{-}\right] \qquad U_{\mu} = i\left[\tilde{U}_{\mu}, K_{-}\right]. \tag{5.14}$$

Commutators $\left[\mathfrak{g}^{-1}, \mathfrak{g}^{+1}\right]$ close into $\mathfrak{g}^{0}$ as follows

$$\begin{aligned}
\left[U_{\mu}, \tilde{U}_{\nu}\right] &= i\eta_{\mu\nu}J_{-} \qquad \left[V^{\mu}, \tilde{V}^{\nu}\right] = i\eta^{\mu\nu}J_{+} \\
\left[V^{\mu}, \tilde{U}_{\nu}\right] &= 2\eta^{\mu\rho}M_{\rho\nu} + i\delta^{\mu}{}_{\nu}\left(J_0 + \Delta\right) \\
\left[U_{\mu}, \tilde{V}^{\nu}\right] &= -2\eta^{\nu\rho}M_{\mu\rho} + i\delta^{\nu}{}_{\mu}\left(J_0 - \Delta\right)
\end{aligned} \tag{5.15}$$

where Δ is the generator that determines the 5-grading

$$\Delta = \frac{1}{2}\left(xp + px\right) \tag{5.16}$$

such that

$$\left[K_{-}, K_{+}\right] = i\Delta \qquad \left[\Delta, K_{\pm}\right] = \pm 2iK_{\pm} \tag{5.17}$$

$$\left[\Delta, U_{\mu}\right] = -iU_{\mu} \quad \left[\Delta, V^{\mu}\right] = -iV^{\mu} \quad \left[\Delta, \tilde{U}_{\mu}\right] = i\tilde{U}_{\mu} \quad \left[\Delta, \tilde{V}^{\mu}\right] = i\tilde{V}^{\mu} \tag{5.18}$$

The quadratic Casimir operators of subalgebras $\mathfrak{so}\left(p,q\right)$, $\mathfrak{sp}\left(2,\mathbb{R}\right)_J$ of grade zero subspace and $\mathfrak{sp}\left(2,\mathbb{R}\right)_K$ generated by $K_{\pm}$ and Δ are

$$\begin{aligned}
M_{\mu\nu}M^{\mu\nu} &= -\mathcal{I}_4 - 2\left(p+q\right) \\
J_{-}J_{+} + J_{+}J_{-} - 2\left(J_0\right)^2 &= \mathcal{I}_4 + \frac{1}{2}\left(p+q\right)^2 \\
K_{-}K_{+} + K_{+}K_{-} - \frac{1}{2}\Delta^2 &= \frac{1}{4}\mathcal{I}_4 + \frac{1}{8}\left(p+q\right)^2
\end{aligned} \tag{5.19}$$

Note that they all reduce to $\mathcal{I}_4$ modulo some additive and multiplicative constants. Noting also that

$$\left(U_{\mu}\tilde{V}^{\mu} + \tilde{V}^{\mu}U_{\mu} - V^{\mu}\tilde{U}_{\mu} - \tilde{U}_{\mu}V^{\mu}\right) = 2\mathcal{I}_4 + \left(p+q\right)\left(p+q+4\right) \tag{5.20}$$

we conclude that there exists a family of degree 2 polynomials in the enveloping algebra of $\mathfrak{so}\,(p+2,q+2)$ that degenerate to a c-number for the minimal unitary realization, in accordance with Joseph's theorem [18]:

$$\begin{aligned} M_{\mu\nu}M^{\mu\nu} &+ \kappa_1\left(J_-J_+ + J_+J_- - 2\,(J_0)^2\right) + 4\kappa_2\left(K_-K_+ + K_+K_- - \frac{1}{2}\Delta^2\right) \\ &- \frac{1}{2}\,(\kappa_1+\kappa_2-1)\left(U_\mu\tilde{V}^\mu + \tilde{V}^\mu U_\mu - V^\mu\tilde{U}_\mu - \tilde{U}_\mu V^\mu\right) \\ &= \frac{1}{2}\,(p+q)\,(p+q-4\,(\kappa_1+\kappa_2)) \end{aligned} \tag{5.21}$$

The quadratic Casimir of $\mathfrak{so}\,(p+2,q+2)$ corresponds to the choice $2\kappa_1 = 2\kappa_2 = -1$ in (5.21). Hence the eigenvalue of the quadratic Casimir for the minimal unitary representation is equal to $\frac{1}{2}\,(p+q)\,(p+q+4)$.

6. Minimal unitary realizations of the quasiconformal groups $\mathrm{SO}^*(2n+4)$

The noncompact group $SO^*(2n+4)$ is a subgroup of $\mathrm{SL}(2n+4,\mathbb{C})$ whose maximal compact subgroup is $\mathrm{U}(n+2)$. We have the inclusions

$$\mathrm{SO}^*(2n+4) \subset \mathrm{SU}^*(2n+4) \subset \mathrm{SL}(2n+4,\mathbb{C}) \tag{6.1}$$

As a matrix group $\mathrm{SU}^*(2n+4)$ is generated by matrices U belonging to $\mathrm{SL}(2n+4,\mathbb{C})$ that satisfy

$$\mathrm{U}\,\mathbb{J} = \mathbb{J}U^* \tag{6.2}$$

where $\mathbb{J}$ is a $(2n+4)\times(2n+4)$ matrix that is antisymmetric

$$\mathbb{J}^T = -\mathbb{J} \tag{6.3}$$

and whose square is the identity matrix

$$\mathbb{J}^2 = -\mathbb{I} \tag{6.4}$$

The matrices U belonging to the subgroup $SO^*(2n+4)$ of $SU^*(2n+4)$ satisfy, in addition, the condition

$$\mathrm{U}\,\mathrm{U}^T = \mathbb{I} \tag{6.5}$$

Thus $\mathrm{SO}^*(2n+4)$ leaves invariant both the Euclidean metric δ_{IJ} and the complex structure $\mathbb{J}_{IJ} = -\mathbb{J}_{JI}$ where $I,J,\ldots = 1,2,\ldots 2n+4$. Hence $\mathrm{SO}^*(2n+4)$ is also a subgroup of the complex rotation group $\mathrm{SO}(2n+4,\mathbb{C})$.

To obtain the 5-grading of the Lie algebra of $\mathrm{SO}^*(2n+4)$ so as to construct its minimal unitary representation we need to consider its decomposition with respect to its subgroup

$$\mathrm{SO}^*(2n)\times\mathrm{SO}^*(4) \subset \mathrm{SO}^*(2n+4) \tag{6.6}$$

where

$$\mathrm{SO}^*(4) = \mathrm{SU}(2)\times\mathrm{SL}(2,\mathbb{R}) \tag{6.7}$$

The distinguished $\mathrm{SL}(2,\mathbb{R})$ subgroup can then be identified with the factor $\mathrm{SL}(2,\mathbb{R})$ above. The corresponding 5-grading of the Lie algebra of $SO^*(2n+4)$ is then

$$\mathfrak{so}^*(2n+4) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus (\mathfrak{so}^*(2n) \oplus \mathfrak{su}(2) \oplus \Delta) \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} \tag{6.8}$$

where grade ± 1 subspaces transform in the $[2n, 2]$ dimensional representation of $\mathrm{SO}^*(2n) \times \mathrm{SU}(2)$. Let X^μ and P_μ be canonical coordinates and momenta in $\mathbb{R}^{(2n)}$:

$$[X^\mu, P_\nu] = i\delta^\mu_\nu \tag{6.9}$$

They satisfy the Hermiticity conditions

$$(X^\mu)^\dagger = \mathbb{J}_{\mu\nu} X^\nu \tag{6.10}$$

$$(P_\mu)^\dagger = \mathbb{J}^{\mu\nu} P_\nu \tag{6.11}$$

Let x be an additional "cocycle" coordinate and p be its conjugate momentum:

$$[x, p] = i \tag{6.12}$$

as in the previous section. The subgroup H is now $SO^*(2n) \times \mathrm{SU}(2)_J$ whose generators are $(M_{\mu\nu},\, J_\pm,\, J_0)$. The grade -1 generators will be denoted as $(U_\mu,\, V^\mu)$ and the grade -2 generator as K_- as in the previous section. The generators of H, its 4-th order invariant $\mathcal{I}_4$ and the negative grade generators are realized as follows:

$$\begin{aligned}
M_{\mu\nu} &= i\delta_{\mu\rho} X^\rho P_\nu - i\delta_{\nu\rho} X^\rho P_\mu & J_0 &= \frac{1}{2}\left(X^\mu P_\mu + P_\mu X^\mu\right) \\
U_\mu &= x P_\mu \qquad V^\mu = x X^\mu & J_- &= X^\mu X^\mu \\
K_- &= \frac{1}{2} x^2 & J_+ &= P_\mu P_\mu \\
\mathcal{I}_4 &= \left(X^\mu X^\mu\right)\left(P_\nu P_\nu\right) + \left(P_\mu P_\mu\right)\left(X^\nu X^\nu\right) \\
&\quad - \left(X^\mu P_\mu\right)\left(P_\nu X^\nu\right) - \left(P_\mu X^\mu\right)\left(X^\nu P_\nu\right)
\end{aligned} \tag{6.13}$$

where $\delta_{\mu\nu}$ is the flat Euclidean metric in $2n$ dimensions.

It is easy to verify that the generators $M_{\mu\nu}$ and $J_{0,\pm}$ satisfy the commutation relations of $\mathfrak{so}^*(2n) \oplus \mathfrak{su}(2)_J$:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\tau}] &= \delta_{\nu\rho} M_{\mu\tau} - \delta_{\mu\rho} M_{\nu\tau} + \eta_{\mu\tau} M_{\nu\rho} - \eta_{\nu\tau} M_{\mu\rho} \\
[J_0, J_\pm] &= \pm 2i J_\pm \qquad [J_-, J_+] = 4i J_0
\end{aligned} \tag{6.14}$$

under which coordinates X^μ (V^μ) and momenta P^μ (U^μ) transform as vectors of $SO^*(2n)$ and form doublets of $\mathrm{SU}(2)$:

$$\begin{aligned}
[J_0, V^\mu] &= -iV^\mu \quad & [J_-, V^\mu] &= 0 \quad & [J_+, V^\mu] &= -2iU^\nu \\
[J_0, U_\mu] &= +iU_\mu \quad & [J_-, U_\mu] &= 2iV_\mu \quad & [J_+, U_\mu] &= 0
\end{aligned} \tag{6.15}$$

The generators in the subspace $\mathfrak{g}^{-1} \oplus \mathfrak{g}^{-2}$ form a Heisenberg algebra

$$[V^\mu, U_\nu] = 2i\delta^\mu{}_\nu\, K_- \,. \tag{6.16}$$

with K_- playing the role of "$\hbar$".

Using the quartic invariant we define the grade $+2$ generator as

$$K_+ = \frac{1}{2}p^2 + \frac{1}{4\,y^2}\left(\mathcal{I}_4 + \frac{4(n-1)^2+3}{2}\right) \tag{6.17}$$

Then the grade $+1$ generators are obtained by commutation relations

$$\tilde{V}^\mu = -i\,[V^\mu, K_+] \qquad \tilde{U}_\mu = -i\,[U_\mu, K_+] \tag{6.18}$$

which explicitly read as follows

$$\begin{aligned}\tilde{V}^\mu &= pX^\mu + \frac{1}{2}x^{-1}\left(P_\nu X^\lambda X^\rho + X^\lambda X^\rho P_\nu\right)\eta^{\mu\nu}\eta_{\lambda\rho}\\ &\quad - \frac{1}{4}x^{-1}\left(X^\mu\left(X^\nu P_\nu + P_\nu X^\nu\right) + \left(X^\nu P_\nu + P_\nu X^\nu\right)X^\mu\right)\\ \tilde{U}_\mu &= pP_\mu - \frac{1}{2}x^{-1}\left(X^\nu P_\lambda P_\rho + P_\lambda P_\rho X^\nu\right)\eta_{\mu\nu}\eta^{\lambda\rho}\\ &\quad + \frac{1}{4}x^{-1}\left(P_\mu\left(X^\nu P_\nu + P_\nu X^\nu\right) + \left(X^\nu P_\nu + P_\nu X^\nu\right)P_\mu\right).\end{aligned} \tag{6.19}$$

Then one finds that the generators in $\mathfrak{g}^{+1}\oplus\mathfrak{g}^{+2}$ subspace form an Heisenberg algebra as well

$$\left[\tilde{V}^\mu, \tilde{U}_\nu\right] = 2i\delta^\mu{}_\nu K_+ \qquad V^\mu = i\left[\tilde{V}^\mu, K_-\right] \qquad U_\mu = i\left[\tilde{U}_\mu, K_-\right]. \tag{6.20}$$

Commutators $\left[\mathfrak{g}^{-1}, \mathfrak{g}^{+1}\right]$ close into $\mathfrak{g}^0$ as follows

$$\begin{aligned}\left[U_\mu, \tilde{U}_\nu\right] &= i\eta_{\mu\nu}J_- \qquad \left[V^\mu, \tilde{V}^\nu\right] = i\eta^{\mu\nu}J_+\\ \left[V^\mu, \tilde{U}_\nu\right] &= 2\eta^{\mu\rho}M_{\rho\nu} + i\delta^\mu{}_\nu\left(J_0 + \Delta\right)\\ \left[U_\mu, \tilde{V}^\nu\right] &= -2\eta^{\nu\rho}M_{\mu\rho} + i\delta^\nu{}_\mu\left(J_0 - \Delta\right)\end{aligned} \tag{6.21}$$

where Δ is the generator that determines the 5-grading

$$\Delta = \frac{1}{2}\left(xp + px\right) \tag{6.22}$$

such that

$$[K_-, K_+] = i\Delta \qquad [\Delta, K_\pm] = \pm 2iK_\pm \tag{6.23}$$

$$[\Delta, U_\mu] = -iU_\mu \quad [\Delta, V^\mu] = -iV^\mu \quad \left[\Delta, \tilde{U}_\mu\right] = i\tilde{U}_\mu \quad \left[\Delta, \tilde{V}^\mu\right] = i\tilde{V}^\mu \tag{6.24}$$

The quadratic Casimir operators of subalgebras $\mathfrak{so}^*(2n)$, $\mathfrak{su}(2)_J$ of grade zero subspace and $\mathfrak{sp}(2,\mathbb{R})_K$ generated by $K_\pm$ and Δ are

$$\begin{aligned}M_{\mu\nu}M^{\mu\nu} &= -\mathcal{I}_4 - 2\left(p+q\right)\\ J_-J_+ + J_+J_- - 2\left(J_0\right)^2 &= \mathcal{I}_4 + \frac{1}{2}\left(p+q\right)^2\\ K_-K_+ + K_+K_- - \frac{1}{2}\Delta^2 &= \frac{1}{4}\mathcal{I}_4 + \frac{1}{8}\left(p+q\right)^2\end{aligned} \tag{6.25}$$

Note that they all reduce to $\mathcal{I}_4$ modulo some additive and multiplicative constants. Noting also that

$$\left(U_\mu \tilde{V}^\mu + \tilde{V}^\mu U_\mu - V^\mu \tilde{U}_\mu - \tilde{U}_\mu V^\mu\right) = 2\mathcal{I}_4 + (p+q)(p+q+4) \tag{6.26}$$

we conclude that there exists a family of degree 2 polynomials in the enveloping algebra of $\mathfrak{so}^*(2n+4)$ that degenerate to a c-number for the minimal unitary realization, in accordance with Joseph's theorem [18]:

$$\begin{aligned} M_{\mu\nu}M^{\mu\nu} &+ \kappa_1\left(J_- J_+ + J_+ J_- - 2(J_0)^2\right) + 4\kappa_2\left(K_- K_+ + K_+ K_- - \frac{1}{2}\Delta^2\right) \\ &- \frac{1}{2}(\kappa_1 + \kappa_2 - 1)\left(U_\mu \tilde{V}^\mu + \tilde{V}^\mu U_\mu - V^\mu \tilde{U}_\mu - \tilde{U}_\mu V^\mu\right) \\ &= n\left(2n - 4(\kappa_1 + \kappa_2 - 1)\right) \end{aligned} \tag{6.27}$$

The quadratic Casimir of $\mathfrak{so}^*(2n+4)$ corresponds to the choice $2\kappa_1 = 2\kappa_2 = -1$ in (6.27). Hence the eigenvalue of the quadratic Casimir for the minimal unitary representation is equal to $2n(n+2)$.

7. Minimal unitary realizations of the quasiconformal groups $SU(n+1, m+1)$

The Lie algebra $\mathfrak{su}(n+1, m+1)$ admits the following five graded decomposition with respect to its subalgebra $\mathfrak{su}(n,m)$:

$$\mathfrak{su}(n+1, m+1) = 1 \oplus 2(n+m) \oplus (\mathfrak{su}(n,m) \oplus \mathfrak{u}(1)) \oplus 2(n+m) \oplus 1$$

It is realized by means of $m+n$ pairs of creation and annihilation operators subject to the following Hermiticity condition:

$$(a^p)^\dagger = \eta^{pq} a_q \qquad [a_q, a^p] = \delta^p{}_q\,. \tag{7.1}$$

Following the steps laid down in previous sections we define generators of H as bilinears in creation and annihilation operators

$$J^p{}_q = a^p a_q - \frac{1}{m+n}\delta^p{}_q a^r a_r \tag{7.2a}$$

Negative grade generators are

$$E = \frac{1}{2}x^2 \quad E^p = x a^p \quad E_q = x a_q \tag{7.2b}$$

The quartic invariant $\mathcal{I}_4$ is related to the quadratic Casimir of H simply

$$\mathcal{I}_4 = \frac{2(m+n)}{m+n-1} J^p{}_q J^q{}_p + \frac{1}{2}\left((m+n)^2 - 1\right) \tag{7.2c}$$

where the additive constant was determined such that

$$F = \frac{1}{2}p^2 + \frac{1}{4}\frac{1}{x^2}\mathcal{I}_4 \tag{7.2d}$$

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The positive grade $\mathfrak{g}^{+1}$ generators are then found by commuting F with generators of $\mathfrak{g}^{-1}$

$$F^p = -i\,[E^p, F] \quad F_q = -i\,[E_q, F] \tag{7.2e}$$

The $\mathfrak{u}(1)$ generator of grade 0 subalgebra is also bilinear in oscillators

$$U = \frac{1}{2}\,(a^p a_p + a_p a^p) \tag{7.2f}$$

Quadratic Casimir of the algebra in this realization reduces to a c-number

$$\begin{aligned}
\mathcal{C}_2 = &-\frac{1}{6} J^p{}_q J^q{}_p + \frac{1}{12}\Delta^2 - \frac{1}{6}\,(EF + FE) - \frac{1}{12}\frac{m+n+2}{m+n} U^2 \\
&- \frac{i}{12}\,(E_p F^p + F^p E_p - F_p E^p - E^p F_p) \\
= &\frac{1}{24}\,(m+n+2)\,(m+n+1)
\end{aligned} \tag{7.3}$$

Positive and negative grades generators transform in the $(n+m)^{+1} \oplus (\overline{n+m})^{-1} \oplus 1^0$ representation of H and satisfy

$$\begin{aligned}
[J^p{}_q, J^s{}_t] &= \delta^s{}_q J^p{}_t - \delta^p{}_t J^s{}_q \\
[J^p{}_q, E^s] &= \delta^s{}_q E^p - \frac{1}{n+m}\delta^p{}_q E^s \\
[J^p{}_q, F^s] &= \delta^s{}_q F^p - \frac{1}{n+m}\delta^p{}_q F^s \\
[J^p{}_q, E_s] &= -\delta^p{}_s E_q + \frac{1}{n+m}\delta^p{}_q E_s \\
[J^p{}_q, F_s] &= -\delta^p{}_s F_q + \frac{1}{n+m}\delta^p{}_q F_s
\end{aligned} \tag{7.4}$$

$$[U, E^p] = E^p \quad [U, F^p] = F^p \quad [U, E_p] = -E_p \quad [U, F_p] = -F_p \tag{7.5}$$

$$[\Delta, E^p] = -iE^p \quad [\Delta, E_p] = -iE_p \quad [\Delta, F^p] = iF^p \quad [\Delta, F_p] = iF_p \tag{7.6}$$

$$[\Delta, F] = 2iF \qquad [\Delta, E] = -2iE \qquad [E, F] = i\Delta \tag{7.7}$$

The remaining non-zero commutation relations are as follows:

$$[E_p, E^q] = 2E \qquad [F_p, F^q] = 2F \tag{7.8}$$

$$\begin{aligned}
[E^p, F_q] &= 2iJ^p{}_q + \frac{m+n+2}{m+n} i\delta^p{}_q U - \delta^p{}_q \Delta \\
[E_p, F^q] &= -2iJ^q{}_p - \frac{m+n+2}{m+n} i\delta^q{}_p U - \delta^p{}_q \Delta
\end{aligned} \tag{7.9}$$

$$[F, E_p] = -iF_p \quad [F, E^p] = -iF^p \quad [E, F_p] = iE_p \quad [E, F^p] = iE^p \tag{7.10}$$

8. Minimal unitary realizations of the quasiconformal groups $SL(n+2,\mathbb{R})$

The construction of the minimal unitary realization of the quasiconformal algebra $\mathfrak{sl}(n+2,\mathbb{R})$ traces the same steps as in the previous section. The five-graded decomposition is as follows

$$\mathfrak{sl}\left(n+2,\mathbb{R}\right) = 1 \oplus \left(n \oplus \tilde{n}\right) \oplus \left(\mathfrak{gl}\left(n,\mathbb{R}\right) \oplus \Delta\right) \oplus \left(n \oplus \tilde{n}\right) \oplus 1$$

Since $n \oplus \tilde{n}$ is a direct sum of two inequivalent self-conjugate vector representations of $\mathfrak{gl}\left(n,\mathbb{R}\right)$, we use coordinates X^{μ} and momenta P_{μ} as oscillator generators, where $\mu = 1,\ldots,n$, with canonical commutation relations:

$$\left[X^{\mu}, P_{\nu}\right] = i\delta^{\mu}{}_{\nu} \tag{8.1}$$

Generators of $\mathfrak{gl}\left(n,\mathbb{R}\right)$

$$\mathcal{L}^{\mu}{}_{\nu} = \frac{i}{2}\left(X^{\mu}P_{\nu} + P_{\nu}X^{\mu}\right) \tag{8.2}$$

have the following commutation relations

$$\left[\mathcal{L}^{\mu}{}_{\nu}, \mathcal{L}^{\tau}{}_{\rho}\right] = \delta^{\tau}{}_{\nu}\mathcal{L}^{\mu}{}_{\rho} - \delta^{\mu}{}_{\rho}\mathcal{L}^{\tau}{}_{\nu} \tag{8.3}$$

The one-dimensional center of the reductive algebra $\mathfrak{gl}\left(n,\mathbb{R}\right)$ is spanned by

$$U = \sum_{\mu=1}^{n} \mathcal{L}^{\mu}{}_{\mu} \tag{8.4}$$

The quadratic Casimir of $\mathfrak{gl}\left(n,\mathbb{R}\right)$ is given simply as a trace of $\mathcal{L}^2$:

$$\mathcal{C}_2\left(\mathfrak{gl}\left(n,\mathbb{R}\right)\right) = \mathcal{L}^{\mu}{}_{\nu}\mathcal{L}^{\nu}{}_{\mu} = \frac{n}{4} - \left(X^{\mu}P_{\mu}\right)\left(P_{\nu}X^{\nu}\right) \tag{8.5}$$

Generators of the negative grades

$$E = \frac{1}{2}x^2 \qquad E^{\mu} = xX^{\mu} \qquad E_{\nu} = xP_{\nu} \tag{8.6}$$

form the Heisenberg algebra

$$\begin{aligned} &\left[E^{\mu}, E_{\nu}\right] = \left(2iE\right)\delta^{\mu}{}_{\nu} \qquad \left[E, E^{\mu}\right] = 0 \\ &\left[E^{\nu}, E^{\mu}\right] = 0 \qquad \left[E, E_{\mu}\right] = 0 \qquad \left[E_{\nu}, E_{\mu}\right] = 0 \end{aligned} \tag{8.7}$$

The generator of grade $+2$ subspace takes on the familiar form

$$F = \frac{1}{2}p^2 + \frac{1}{2}\frac{1}{x^2}\mathcal{I}_4 = \frac{1}{2}p^2 + \frac{1}{2}\frac{1}{x^2}\left(\frac{n^2-1}{4} - \left(X^{\mu}P_{\mu}\right)\left(P_{\nu}X^{\nu}\right)\right) \tag{8.8a}$$

and leads to the following grade $+1$ generators

$$F^{\mu} = -i\left[xX^{\mu}, F\right] = pX^{\mu} - \frac{1}{2}\frac{1}{x}\left(X^{\mu}\left(P_{\nu}X^{\nu}\right) + \left(X^{\nu}P_{\nu}\right)X^{\mu}\right) \tag{8.8b}$$

$$F_\mu = -i\,[xP_\mu, F] = pP_\mu + \frac{1}{2}\frac{1}{x}\left(P_\mu\left(P_\nu X^\nu\right) + \left(X^\nu P_\nu\right)P_\mu\right) \tag{8.8c}$$

They form the dual Heisenberg algebra

$$\begin{aligned} &[F^\mu, F_\nu] = (2iF)\,\delta^\mu{}_\nu \qquad [F, F^\mu] = 0 \\ &[F^\nu, F^\mu] = 0 \qquad [F, F_\mu] = 0 \qquad [F_\nu, F_\mu] = 0 \end{aligned} \tag{8.9}$$

Subspaces $\mathfrak{g}^{\pm 1}$ transform under $\mathfrak{gl}\,(n,\mathbb{R})$ as $\mathbf{n}\oplus\tilde{\mathbf{n}}$ each:

$$\begin{aligned} &[\mathcal{L}^\mu{}_\nu, E^\rho] = \delta^\rho{}_\nu E^\mu \qquad [\mathcal{L}^\mu{}_\nu, E_\rho] = -\delta^\mu{}_\rho E_\nu \\ &[\mathcal{L}^\mu{}_\nu, F^\rho] = \delta^\rho{}_\nu F^\mu \qquad [\mathcal{L}^\mu{}_\nu, F_\rho] = -\delta^\mu{}_\rho F_\nu \end{aligned} \tag{8.10}$$

Other cross grade commutation relations read

$$[E, F^\mu] = iE^\mu \quad [E, F_\mu] = iE_\mu \quad [F, E^\mu] = -iF^\mu \quad [F, E_\mu] = -iF_\mu \tag{8.11}$$

$$[E, F] = i\Delta \qquad [\Delta, E] = -2iE \qquad [\Delta, F] = +2iF \tag{8.12}$$

$$\begin{aligned} [\Delta, E^\mu] = -iE^\mu \qquad [\Delta, E_\mu] = -iE_\mu \\ [\Delta, F^\mu] = +iF^\mu \qquad [\Delta, F_\mu] = +iF_\mu \end{aligned} \tag{8.13}$$

$$[E^\mu, F^\nu] = 0 \qquad [E_\mu, F_\nu] = 0 \tag{8.14}$$

$$\begin{aligned} [E^\mu, F_\nu] = 2\mathcal{L}^\mu{}_\nu + \delta^\mu{}_\nu\,(U + i\Delta) \\ [E_\mu, F^\nu] = 2\mathcal{L}^\nu{}_\mu + \delta^\nu{}_\mu\,(U - i\Delta) \end{aligned} \tag{8.15}$$

A short calculation verifies that the quadratic Casimir of $\mathfrak{sl}\,(n+2,\mathbb{R})$ is

$$\begin{aligned} C_2 = \mathcal{L}^\mu{}_\nu\mathcal{L}^\nu{}_\mu + \frac{1}{2}\left(\mathcal{L}^\mu{}_\mu\right)\left(\mathcal{L}^\nu{}_\nu\right) - \frac{1}{2}\Delta^2 + (EF + FE) \\ + \frac{1}{2}\left(E^\mu F_\mu + F_\mu E^\mu - F^\mu E_\mu - E_\mu F^\mu\right) \end{aligned} \tag{8.16}$$

When evaluated on the quasi-conformal realization it reduces to c-number:

$$C_2 = -\frac{1}{4}\,(n+2)\,(n+1) \tag{8.17}$$

Quadratic Casimir is just an element of the Joseph ideal as follows from relations below

$$\mathcal{L}^\mu{}_\nu\mathcal{L}^\nu{}_\mu + \frac{1}{2}\left(\mathcal{L}^\mu{}_\mu\right)\left(\mathcal{L}^\nu{}_\nu\right) = \frac{3}{2}\mathcal{I}_4 + \frac{1}{8}\left(3 - 2n\,(n-1)\right) \tag{8.18}$$

$$-\frac{1}{2}\Delta^2 + (EF + FE) = \frac{1}{2}\mathcal{I}_4 - \frac{3}{8} \tag{8.19}$$

$$\frac{1}{2}\left(E^\mu F_\mu + F_\mu E^\mu - F^\mu E_\mu - E_\mu F^\mu\right) = -2\mathcal{I}_4 - n + \frac{1}{2} \tag{8.20}$$

Namely, any linear combination of the above three expressions collapses to a c-number provided coefficients are matched as to cancel all $\mathcal{I}_4$.

9. Minimal unitary realizations of Lie superalgebras

In this section we will extend the construction of the minimal unitary representations of Lie groups obtained by quantization of their quasi conformal realizations to the construction of the minimal representations of Lie superalgebras. In analogy with the Lie algebras we consider 5-graded simple Lie superalgebras

$$\mathfrak{g}_B^{-2} \oplus \mathfrak{g}_F^{-1} \oplus \left(\mathfrak{g}^0 \oplus \Delta\right)_B \oplus \mathfrak{g}_F^{+1} \oplus \mathfrak{g}_B^{+2} \tag{9.1}$$

where $\mathfrak{g}^{\pm 2}$ are 1-dimensional subspaces each, and $\mathfrak{g}^{-2} \oplus \Delta \oplus \mathfrak{g}^{+2}$ form $\mathfrak{sl}(2, \mathbb{R})$. In this paper we restrict ourselves to Lie superalgebras whose grade ± 1 generators are all odd (exhaustively). Let J^a denote generators of $\mathfrak{g}^0$

$$\left[J^a , J^b\right] = f^{ab}{}_c J^c \tag{9.2}$$

and let ρ denote the irreducible orthogonal representation with a definite Dynkin index by which $\mathfrak{g}^0$ acts on $\mathfrak{g}^{\pm 1}$

$$[J^a , E^\alpha] = \lambda^{a\alpha}{}_\beta E^\beta \qquad [J^a , F^\alpha] = \lambda^{a\alpha}{}_\beta F^\beta \tag{9.3}$$

where E^α are odd generators that span $\mathfrak{g}^{-1}$

$$\left\{E^\alpha , E^\beta\right\} = \Omega_s^{\alpha\beta} E \tag{9.4}$$

and F^α generators that span $\mathfrak{g}^{+1}$

$$\left\{F^\alpha , F^\beta\right\} = \Omega_s^{\alpha\beta} F \tag{9.5}$$

and $\Omega_s^{\alpha\beta}$ is now a symmetric invariant tensor. Hence negative (positive) grade subspace form a super Heisenberg algebra. Due to 5-graded structure we can impose

$$F^\alpha = [E^\alpha , F] \qquad E^\alpha = [E , F^\alpha] \tag{9.6}$$

Now we realize the generators using anti-commuting covariant oscillators ξ^α

$$\left\{\xi^\alpha , \xi^\beta\right\} = \Omega_s^{\alpha\beta} \tag{9.7}$$

plus an extra bosonic coordinate y and its conjugate momentum p. The non-positive grade generators take the form [5]

$$E = \frac{1}{2} y^2 \qquad E^\alpha = y \xi^\alpha \qquad J^a = -\frac{1}{2} \lambda^a{}_{\alpha\beta} \xi^\alpha \xi^\beta \tag{9.8}$$

The quadratic Casimir of $\mathfrak{g}^0$ is taken to be

$$\mathcal{C}_2\left(\mathfrak{g}^0\right) = \eta_{ab} J^a J^b \tag{9.9}$$

[5]We are following conventions of [21] for superalgebras as well.

and the grade +2 generator F is assumed to be of the form

$$F = \frac{1}{2}p^2 + \kappa y^{-2}\left(\mathcal{C}_2 + \mathfrak{C}\right) \tag{9.10}$$

for some constants κ and $\mathfrak{C}$ to be determined later. Commuting E with F^α we obtain

$$F^\alpha = ip\,\xi^\alpha + \kappa y^{-1}\left[\xi^\alpha\,,\mathcal{C}_2\right] \tag{9.11}$$

By inspection we have

$$\left[\xi^\alpha\,,\mathcal{C}_2\right] = -2\left(\lambda^a\right)^\alpha{}_\beta \xi^\beta J_a + C_\rho\,\xi^\alpha \tag{9.12}$$

and we shall use the following Ansatz [21]

$$\left(\lambda^a\right)^\beta{}_\gamma (\lambda_a)^\alpha{}_\delta + \left(\lambda^a\right)^{\beta\alpha}(\lambda_a)_{\gamma\delta} = \frac{C_\rho}{N-1}\left(\Omega_s^{\alpha\beta}\Omega_{s\gamma\delta} + \delta^\beta{}_\gamma\delta^\alpha{}_\delta - 2\delta^\alpha{}_\gamma\delta^\beta{}_\delta\right) \tag{9.13}$$

to calculate the remaining super commutation relations.

For the anticommutators of grade +1 generators with grade -1 generators we get

$$\begin{aligned}\left\{E^\alpha\,,F^\beta\right\} &= i\,(y\,p)\,\Omega_s^{\alpha\beta} + \xi^\beta\,\xi^\alpha + \kappa\left\{\xi^\alpha\,,\left[\xi^\beta\,,\mathcal{C}_2\right]\right\} \\ \left\{E^\alpha,\,F^\beta\right\} &= -\Omega_s^{\alpha\beta}\Delta - 6\kappa\left(\lambda^a\right)^{\beta\alpha}J_a + \left(\frac{3\kappa C_\rho}{N-1} - \frac{1}{2}\right)\left(\xi^\beta\xi^\alpha - \xi^\alpha\xi^\beta\right).\end{aligned} \tag{9.14}$$

Now the bilinears $\left(\xi^\beta\xi^\alpha - \xi^\alpha\xi^\beta\right)$ on the right hand side generate the Lie algebra $\mathfrak{so}(N)$. Therefore, for those Lie superalgebras whose grade zero subalgebras g^0 are different from $\mathfrak{so}(N)$ we must impose the constraint:

$$\left(\frac{3\kappa C_\rho}{N-1} - \frac{1}{2}\right) \tag{9.15}$$

For the anticommutators $\left\{F^\alpha,\,F^\beta\right\}$ we get

$$\left\{F^\alpha\,,F^\beta\right\} = -p^2\Omega_s^{\alpha\beta} - \frac{\kappa}{y^2}\left(\xi^\alpha\left[\xi^\beta\,,\mathcal{C}_2\right] + \xi^\beta\left[\xi^\alpha\,,\mathcal{C}_2\right] - \kappa\left\{\left[\xi^\alpha\,,\mathcal{C}_2\right],\left[\xi^\beta\,,\mathcal{C}_2\right]\right\}\right) \tag{9.16a}$$

$$\begin{aligned}\left\{F^\alpha,\,F^\beta\right\} &= -2F\Omega_s^{\alpha\beta} = -2\left(\frac{p^2}{2} + \frac{k}{x^2}\left(\frac{1}{2}\kappa C_\rho^2 + \frac{1}{2}C_\rho\right)\right)\Omega_s^{\alpha\beta} \\ &\quad - \frac{\kappa}{x^2}\left(-4\left(\xi^\alpha(\lambda_a)^\beta{}_\gamma + \xi^\beta(\lambda_a)^\alpha{}_\gamma\right)\xi^\gamma J^a + 12\kappa\left(\lambda_a\lambda_b\right)^{\alpha\beta}J^bJ^a\right. \\ &\quad \left. +8\kappa\left(\lambda_a\lambda_b\right)^{\beta\alpha}J^bJ^a - 4\kappa(\lambda_a)^\alpha{}_\delta(\lambda_b)^\beta{}_\gamma\xi^\delta\xi^\gamma f^{ab}{}_cJ^c\right)\end{aligned} \tag{9.16b}$$

Taking the Ω_s trace we obtain

$$N = 8 - 10\kappa i_\rho \ell^2 + 2\kappa C_{\rm adj} \qquad\qquad 2\mathfrak{C} = \kappa C_\rho^2 + C_\rho \tag{9.17}$$

Taking into account that

$$\frac{i_\rho\ell^2}{C_\rho} = \frac{N}{D} \qquad C_{\rm adj} = +\ell^2 h^\vee \tag{9.18}$$

and using (9.15) we obtain the following constraint equation for super quasiconformal algebras, whose grade zero algebras are different from $\mathfrak{so}(N)$:

$$\frac{h^\vee}{i_\rho} = 5 + \frac{3D}{N(N-1)}\,(N-8)\,. \tag{9.19}$$

Now we also have

$$[F,\, F^\alpha] = \frac{\kappa}{x^3}\,((\mathcal{C}_2 + \mathfrak{C})\,\xi^\alpha + \xi^\alpha\,(\mathcal{C}_2 + \mathfrak{C}) + \kappa\,[\mathcal{C}_2, [\xi^\alpha,\, \mathcal{C}_2]]) \tag{9.20}$$

Hence the constraint imposed by the commutation relation $[F,\, F^\alpha] = 0$ is

$$2\xi^\alpha\,(\mathcal{C}_2 + \mathfrak{C}) - C_\rho \xi^\alpha + (2 + 4\kappa C_\rho)\,(\lambda_a)^\alpha{}_\beta - \kappa C_\rho^2 \xi^\alpha - 4\kappa(\lambda_a \lambda_b)^\alpha{}_\beta \xi^\beta J^b J^a = 0 \tag{9.21}$$

which, upon contraction with $\xi^\gamma \Omega_{s\gamma\alpha}$ leads to the following condition

$$\frac{h^\vee}{i_\rho} = -1 + \frac{D}{N\,(N-1)}\,(N+8)\,. \tag{9.22}$$

These two conditions (9.19) and (9.22) agree provided

$$D = \frac{3N\,(N-1)}{16-N} \tag{9.23}$$

which is also the condition for them to agree with the equation

$$h^\vee = 2i_\rho \left(\frac{D}{N} + \frac{3D}{N(1-N)} + 1 \right) \tag{9.24}$$

which was obtained as a consistency condition for the existence of certain class of infinite dimensional nonlinear superconformal algebras [21, 20].

Looking at the tables of simple Lie superalgebras [21, 20] consistent with our Ansatz we find the following simple Lie algebras $\mathfrak{g}_0$ and their irreps of dimension N that satisfy these conditions:

$\mathfrak{g}_0$	D	N
$\mathfrak{b}_3$	21	8_s
$\mathfrak{g}_2$	14	7

(9.25)

These solutions correspond to the Lie superalgebras $\mathfrak{f}(4)$ with even subalgebra $\mathfrak{b}_3 \oplus \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{g}(3)$ with even subalgebras $\mathfrak{g}_2 \oplus \mathfrak{sl}(2,\mathbb{R})$.

We should note that the real forms with an even subgroup of the form $H \times \mathrm{SL}(2,\mathbb{R})$, with H simple, admit unitary representations only if H is compact.

10. Minimal representations of $OSp\,(N|2,\mathbb{R})$ and $D(2,1;\alpha)$

10.1 $O\,\mathrm{Sp}(N|2,\mathbb{R})$

For the Lie superalgebras $\mathfrak{osp}(N|2,\mathbb{R})$ the constraint equation (9.15) does not follow from the commutation relations and hence must not be imposed. In this case the bilinears $(\xi^\beta\xi^\alpha - \xi^\alpha\xi^\beta)$ generate the Lie algebra of the even subgroup $\mathrm{SO}(N)$ (super analog of

$\mathrm{Sp}(2n,\mathbb{R})$. This realization corresponds to the singleton representation of $\mathrm{SO}(N)$ and its quadratic Casimir is a c-number. As a consequence of this the Jacobi identities require either that we set $\kappa = 0$ as in the case of the minimal realization of $\mathrm{Sp}(2n+2,\mathbb{R})$. Hence the minimal realization reduces to a realization in terms of bilinears of fermionic and bosonic oscillators. Thus the minimal unitary representations of $O\,\mathrm{Sp}(N|2,\mathbb{R})$ are simply the supersingleton representations. The singleton supermultiplets of $O\,\mathrm{Sp}(N|2,\mathbb{R})$ were studied in [27].

10.2 $D(2,1;\sigma)$

There is a one parameter family of simple Lie superalgebras of the same dimension that has no analog in the theory of ordinary Lie algebras. It is the family $D\,(2,1;\sigma)$. The real forms of interest to us that admit unitary representations has the even subgroup $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SL}(2,\mathbb{R})$. It has a five grading of the form

$$D\,(2,1;\sigma) = \mathbf{1} \oplus (\mathbf{2},\mathbf{2}) \oplus (\mathfrak{su}\,(2) \oplus \mathfrak{su}\,(2) \oplus \Delta) \oplus (\mathbf{2},\mathbf{2}) \oplus \mathbf{1} \tag{10.1}$$

Let $X^{\alpha,\dot\alpha}$ be 4 fermionic oscillators with canonical anti-commutation relations:

$$\left\{X^{\alpha,\dot\alpha}, X^{\beta,\dot\beta}\right\} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} \tag{10.2}$$

where $\alpha,\dot\alpha,\ldots$ denote the spinor indices of the two $\mathrm{SU}(2)$ factors. Let

$$E = \frac{1}{2}x^2 \qquad E^{\alpha,\dot\alpha} = xX^{\alpha,\dot\alpha} \qquad \Delta = \frac{1}{2}\,(xp+px) \tag{10.3}$$

and take the generators of $\mathfrak{g}^0$ to be of the form

$$\begin{aligned} M_{(1)}^{\alpha,\beta} &= \frac{1}{4}\epsilon_{\dot\alpha\dot\beta}\left(X^{\alpha,\dot\alpha}X^{\beta,\dot\beta} + X^{\beta,\dot\beta}X^{\alpha,\dot\alpha}\right) \\ M_{(2)}^{\dot\alpha,\dot\beta} &= \frac{1}{4}\epsilon_{\alpha\beta}\left(X^{\alpha,\dot\alpha}X^{\beta,\dot\beta} + X^{\beta,\dot\beta}X^{\alpha,\dot\alpha}\right) \end{aligned} \tag{10.4}$$

They satisfy commutation relations of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$

$$\begin{aligned} \left[M_{(1)}^{\alpha,\beta}, M_{(1)}^{\lambda,\mu}\right] &= \epsilon^{\lambda\beta}M_{(1)}^{\alpha,\mu} + \epsilon^{\mu\alpha}M_{(1)}^{\beta,\lambda} \\ \left[M_{(2)}^{\dot\alpha,\dot\beta}, M_{(2)}^{\dot\lambda,\dot\mu}\right] &= \epsilon^{\dot\lambda\dot\beta}M_{(2)}^{\dot\alpha,\dot\mu} + \epsilon^{\dot\mu\dot\alpha}M_{(2)}^{\dot\beta,\dot\lambda} \\ \left[M_{(1)}^{\alpha,\beta}, M_{(2)}^{\dot\lambda,\dot\mu}\right] &= 0 \end{aligned} \tag{10.5}$$

Their quadratic Casimirs are

$$\mathcal{I}_4 = \epsilon_{\alpha\beta}\epsilon_{\lambda\mu}M_{(1)}^{\alpha\lambda}M_{(1)}^{\beta\mu} \qquad \mathcal{J}_4 = \epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\lambda\dot\mu}M_{(2)}^{\dot\alpha\dot\lambda}M_{(2)}^{\dot\beta\dot\mu} \tag{10.6}$$

Their sum is a c-number $\mathcal{I}_4 + \mathcal{J}_4 = -\frac{3}{2}$. We use just one to construct generator of $\mathfrak{g}^{+2}$

$$F = \frac{1}{2}p^2 + \frac{\sigma}{x^2}\left(\mathcal{I}_4 + \frac{3}{4} + \frac{9}{8}\sigma\right) \tag{10.7}$$

and

$$F^{\alpha\dot{\alpha}} = -i\left[E^{\alpha\dot{\alpha}}, F\right] \tag{10.8}$$

Then

$$\left[F^{\alpha\dot{\alpha}}, F^{\beta\dot{\beta}}\right] = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}F \qquad \left[F^{\alpha\dot{\alpha}}, F\right] = 0 \tag{10.9}$$

Also

$$\left[F^{\alpha\dot{\alpha}}, E^{\beta\dot{\beta}}\right] = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\Delta - (1-3\sigma)\, i\epsilon^{\alpha\beta}M_{(2)}^{\dot{\alpha}\dot{\beta}} - (1+3\sigma)\, i\epsilon^{\dot{\alpha}\dot{\beta}}M_{(1)}^{\alpha\beta} \tag{10.10}$$

The parameter σ is left undetermined by the Jacobi identities. For $\sigma = 0$ the superalgebra $D(2,1,\sigma)$ is isomorphic to $O\,\mathrm{Sp}(4|2,\mathbb{R})$ and for the values $\sigma = \pm\frac{1}{3}$ it reduces to

$$SU(2|1,1) \times SU(2)\,.$$

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